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ELASTOTHERMODYNAMIC DAMPING IN LAMINATED COMPOSITES

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Abstract—When a composite material is subjected to a time-harmonic stress field (homogeneous or inhomogeneous), different phases undergo different temperature fluctuations due to the well-known thermoelastic effect. As a result irreversible heat conduction occurs within each phase and between phases, and entropy is produced. This entropy production is the genesis of elastothermodynamic damping, and manifests itself as a conversion of work into heat. This is one of a large number of sources of damping in a real composite.

Mechanics-based analytical methods for predicting damping due to various relaxation mechanisms have not been reported. Therefore, it becomes difficult to divide the total (experimentally measured) damping into its various constituents. This defines the objective of the present work: we present a methodology for predicting the elastothermodynamic damping of an N-layer laminated composite. The stress field may be quite general so long as the resulting heat conduction occurs only in the direction orthogonal to the laminae. By way of illustration, numerical results are presented for a symmetric three-layer plate in biaxial bending. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Damping is a critically important property from the viewpoint of vibration suppression in a variety of applications: submarines, aircraft structures, automobile bodies, and particularly in large and flexible aerospace structures, e.g., the space station. The extensive use of laminated composites in these structures is well known. Therefore, there is a need for the design and manufacture of laminated composites with high damping. For a given material there are many mechanisms contributing to the total damping, e.g., point defect relaxation, dislocation motion, grain boundary sliding, coulomb friction at the inclusion-matrix interface, magnetoelastic effects, and elastothermodynamic effects (Nowick and Berry, 1972; De Batist, 1972; Kinra and Wolfenden, 1992; Wolfenden and Wolla, 1992). Most of these damping mechanisms offer a significant damping only over a small range of frequency, temperature, or stress, and are, in general, nonlinear (i.e., damping depends upon the stress amplitude). Elastothermodynamic‡ damping (also known as thermoelastic damping) offers an attractive alternative to these sources of damping. In this case the damping is due to thermal currents within the structure and is not accompanied by a loss of stiffness or strength. Moreover, elastothermodynamic damping is linear, i.e., it is independent of the stress amplitude. Finally, it is envisioned that the elastothermodynamic damping of composite materials and structures can be tailored in a manner analogous to that used currently for tailoring stiffness, strength, coefficient of thermal expansion, etc.

Whenever a material is stressed in a reversible adiabatic (i.e., isentropic) process, there is always a change in its temperature, however small. This phenomenon is known as the thermoelastic effect and can be deduced from the Maxwell's relations (Zemansky and Dittman, 1981); for an isotropic material

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‡ We propose that *elastothermodynamic* is a more appropriate qualifier for the word damping than *thermoelastic*. Historically (Boley and Weiner, 1960), the word *thermoelastic* has been used to connote a situation where the cause is an externally applied temperature field (*thermo*), and the effect is an elastic stress field (*elastic*), hence *thermoelastic*. In the present context of damping, there is no externally applied temperature field. Instead, the cause is an externally applied traction field and the accompanying stress field (*elasto*), and the effect is a temperature field (*thermo*). Moreover, the dissipation of mechanical energy occurs only if the stress field is time dependent (*dynamic*), hence *elastothermodynamic*.

$$\left(\frac{\partial T}{\partial \sigma_{kk}}\right)_S = -T\frac{\alpha}{C} \tag{1}$$

where T is the absolute temperature, σ_{kk} is the hydrostatic stress, α is the linear coefficient of thermal expansion, C is the specific heat *per unit volume*, and the subscript S denotes an isentropic process. Note that for isotropic materials, the application of a shear stress does not produce a change in temperature; hence the presence of σ_{kk} in (1). Since the temperature and stress fields are coupled, inhomogeneities in stress (e.g., stresses due to bending) and material properties (e.g., a laminated plate) result in inhomogeneities in temperature. Heat conducts from the high-temperature regions to the low-temperature regions and, as a consequence of the second law of thermodynamics, entropy is produced which is manifested as a conversion of useful mechanical energy into heat; we define this process as elastothermodynamic damping (entropic description). An alternative yet equivalent view of this phenomenon is as follows. The fluctuating temperature produces a thermal strain field that is out of phase with the applied stress field. This phase difference may be used to calculate the conversion of useful mechanical energy into heat; we define this as the mechanical description of elastothermodynamic damping. Bishop and Kinra (1995b) have recently proven the equivalence of the two descriptions for a general composite material with anisotropic (or triclinic) phases and with thermally perfect or imperfect interfaces between the constituent phases.

Following Zener's seminal work (1937, 1938), several researchers have investigated thermoelastic damping over the past sixty years in *homogeneous* structures (Eshelby, 1949; Biot, 1956; Lücke, 1956; Deresiewicz, 1957; Alblas, 1961, 1981; Chadwick, 1962; Goodman, 1962; Tasi, 1963; Tasi and Herrmann, 1964; Gillis, 1968; Shieh, 1971, 1975, 1979; Lee, 1985, and Landau and Lifshitz, 1986). Recently, Kinra and his coworkers have studied elastothermodynamic damping in cracked media (Kinra and Bishop, 1996), and in lamellar, fibrous, and particulate composite materials (Kinra and Milligan, 1994; Milligan and Kinra, 1993, 1995a,b; Bishop and Kinra, 1993, 1994, 1995, 1996).

The total damping of a composite is due to a large number of relaxation mechanisms (see, for example, Nowick and Berry (1972)). Much of the damping research to date has been empirical or semi-empirical in nature. So far as we know, mechanics-based analytical methods for *predicting* the damping due to a particular relaxation mechanism have not been reported. In the absence of such analytical methods, it becomes difficult to divide the total damping (measured experimentally) into its various constituents, and to *design* composite materials with high damping (or, occasionally, with low damping for certain applications). This defines the objective of the present work. In this paper we present an analysis of the elastothermodynamic damping of an N-layer laminated composite with thermally perfect or imperfect interfaces. The stress field may be quite general so long as the resulting heat conduction occurs only in the thickness direction. As an illustrative example of the general procedure reported in this paper, results are presented for a symmetric three-layer plate subjected to a time-harmonic, biaxial bending with thermally *perfect* interfaces. We will show that by a proper choice of constituent materials, lamina size, and frequency, one can design laminated composites such that the contribution of the elastothermodynamic damping is a significant fraction of the total damping.

2. GENERAL THEORY

The field equations required for calculating the entropy produced and external work lost in an isotropic homogeneous thermoelastic material are now assembled. The temperature field is governed by the *two-way coupled* coupled heat conduction equation (Nowinski, 1978)

$$C\frac{\partial T}{\partial t} = k\nabla^2 T - \alpha T\frac{\partial \sigma_{kk}}{\partial t} \tag{2}$$

where t is time, and k is the thermal conductivity. The last term is due to the thermoelastic effect, (1). The qualifier "two-way" implies that the stress field affects the temperature field, and conversely the temperature field affects the stress field. For most crystalline solids the relative temperature change resulting from the thermoelastic effect is very small. For example, if an initially unstressed rod of steel at an equilibrium temperature of $T_o = 300$ K is subjected to a uniaxial tensile stress comparable to its yield stress, say 700 MPa, the decrease in temperature is 0.7 K, i.e., $|(T-T_o)/T_o| = 2\times 10^{-3}$. It is well known that the thermal stresses produced by these small changes in temperature are negligibly small compared to the applied stresses (Boley and Weiner, 1960). Therefore, for our purpose of calculating the small changes in temperature resulting from the thermoelastic effect, we assume that the stress field is independent of the temperature field. Equation (2) now becomes the *linear one-way* coupled heat conduction equation. Moreover, without introducing an appreciable error, the algebraic effort can be greatly reduced by replacing T with T_o in the last term (Nowinski, 1978); then (2) reduces to

$$C\frac{\partial T}{\partial t} = k\nabla^2 T - \alpha T_o \frac{\partial \sigma_{kk}}{\partial t}. \tag{3}$$

In the remainder of this paper the attention is confined to time-harmonic forced vibrations with a circular frequency, ω. Then the stress field may be written as,

$$\sigma_{kk}(\mathbf{x}, t) = \sigma_{kk}(\mathbf{x})e^{i\omega t} \tag{4}$$

where $\mathbf{x}$ is the spatial coordinate vector. Here, $\sigma_{kk}(\mathbf{x})$ is taken to be real-valued, and as usual, a real-valued stress is obtained by taking either the real or imaginary part of $\sigma_{kk}(\mathbf{x}, t)$. With reference to (3), since σ_{kk} is now time-harmonic the fluctuations in T will necessarily be time-harmonic. Introducing a complex equilibrium temperature $T_o^* = T_o(1+i)$, the temperature field may be written as

$$T(\mathbf{x}, t) = T_o^* + V(\mathbf{x}, \omega)e^{i\omega t}. \tag{5}$$

The term $Ve^{i\omega t}$ represents the fluctuations of temperature about the equilibrium temperature T_o^*. The following identity due to Brillouin (1953) will be found quite useful for obtaining quantities time-averaged over one cycle,

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{Re}(F)\,\mathrm{Re}(f)\,\mathrm{d}t = \frac{1}{2}\mathrm{Re}(F\bar{f}) = \frac{1}{2}\mathrm{Re}(\bar{F}f) \tag{6}$$

where F and f are two time-harmonic complex quantities with a frequency ω, and the overbar ($\bar{\ }$) denotes the complex conjugate.

An expression is now derived for the entropy produced per unit volume per cycle of vibration, Δs, in terms of V. The rate of entropy produced per unit volume, $\dot{s}_p$, due to irreversible heat conduction is given by (de Groot and Mazur, 1984)

$$\dot{s}_p = \frac{k}{T^2}\nabla T\cdot\nabla T \tag{7}$$

where the symbol $\cdot$ denotes the dot product of two vectors and the dot over a symbol ($\dot{\ }$) denotes partial differentiation with respect to time. Consistent with the assumptions of (3), (7) is now approximated by replacing T with T_o in the denominator, then

$$\dot{s}_p = \frac{k}{T_o^2}\nabla T\cdot\nabla T. \tag{8}$$

The entropy produced per unit volume during one cycle of vibration is given by

$$\Delta s = \oint \dot{s}_p \, \mathrm{d}t. \tag{9}$$

Substituting the real part of the temperature from (5) into (8) and using (9) and (6) yields

$$\Delta s(x, \omega) = \frac{k}{T_o^2} \frac{\pi}{\omega} \mathrm{Re}(\nabla V \cdot \overline{\nabla V}). \tag{10}$$

An additional amount of entropy, $\Delta S^{interface}$, is produced if the medium has any thermally imperfect interfaces at which the temperature is discontinuous (Bishop and Kinra, 1996). An expression for $\Delta S^{interface}$ will be presented in Section 3.2. Thus, the total entropy produced per cycle, ΔS, over the entire body occupying a region R is given by

$$\Delta S(\omega) = \int_R \Delta s \, \mathrm{d}R + \Delta S^{interface}. \tag{11}$$

An expression is now derived for the work lost per unit volume per cycle, Δw, in terms of V. The work lost per unit volume during a cycle of vibration is given by (Nowick and Berry, 1972, Ch. 1.3)

$$\Delta w = \oint \sigma_{lm} \, \mathrm{d}\varepsilon_{lm} \tag{12}$$

where σ_{lm} is the stress tensor, and ε_{lm} is the strain tensor. The change in temperature due to the thermoelastic effect produces a thermal strain. The strain tensor ε_{lm} may be decomposed into its elastic part ε_{lm}^{el} and its thermal part ε_{lm}^{th}

$$\varepsilon_{lm} = \varepsilon_{lm}^{el} + \varepsilon_{lm}^{th}. \tag{13}$$

The elastic part of the strain field does not contribute to the work lost since it is always in phase with the stress field. Furthermore, the time-harmonic thermal strain may be written as

$$\varepsilon_{lm}^{th}(\mathbf{x}, t) = \varepsilon_{lm}^{th}(\mathbf{x}, \omega) e^{i\omega t}. \tag{14}$$

Taking the real values of (4) and (14), substituting into (12), and using (6) yields

$$\Delta w(\mathbf{x}, \omega) = -\pi \sigma_{lm}(\mathbf{x}) \, \mathrm{Im}(\varepsilon_{lm}^{th}(\mathbf{x}, \omega)). \tag{15}$$

The thermal strain is given by

$$\varepsilon_{lm}^{th}(\mathbf{x}, \omega) = \alpha \delta_{lm} V(\mathbf{x}, \omega) \tag{16}$$

where δ_{lm} is the Kronecker delta. Substituting (16) into (15) gives

$$\Delta w(\mathbf{x}, \omega) = -\pi \alpha \sigma_{kk}(\mathbf{x}) \, \mathrm{Im} \, V(\mathbf{x}, \omega). \tag{17}$$

The total work lost over the entire body occupying a region R is

$$\Delta W(\omega) = \int_R \Delta w(\mathbf{x}, \omega) \, \mathrm{d}R. \tag{18}$$

Recently, we considered elastothermodynamic damping in a general composite material subjected to a steady-state time-harmonic external loading (Bishop and Kinra, 1996). Each of the constituent phases was taken to be generally anisotropic. The interfaces between the phases could be thermally perfect or imperfect. The outer boundaries of the composite were assumed to be adiabatic. As expected (Bejan, 1996), we verified that *over one cycle*, the *total* work lost throughout the composite, ΔW, is equal to the *total* entropy produced throughout the composite, ΔS, times the equilibrium temperature, T_o, i.e.,

$$\Delta W = T_o \Delta S. \tag{19}$$

However, contrary to intuition, we found that $\Delta w \neq T_o \Delta s$ pointwise, i.e., the work lost at a point is not necessarily equal to the entropy produced at that point times the equilibrium temperature. Following Zener (1938), $T_o \Delta S$ will be referred to as the total heat increment per cycle.

Finally, a specific damping capacity for the entire laminate, Ψ, is defined as the total work lost during one cycle normalized by the maximum elastic energy stored in the composite during that cycle, W,

$$\Psi = \frac{\Delta W}{W} \tag{20}$$

where

$$W = \int_R \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}R. \tag{21}$$

3. ELASTOTHERMODYNAMIC ANALYSIS OF A LAMINATED COMPOSITE

We now apply the general theory of Section 2 to the special case of laminated composites.

3.1. *Temperature*

The temperature field is now solved for an N-layer laminated composite with perfect or imperfect thermal interfaces subjected to a time-harmonic stress field (see Fig. 1). Each

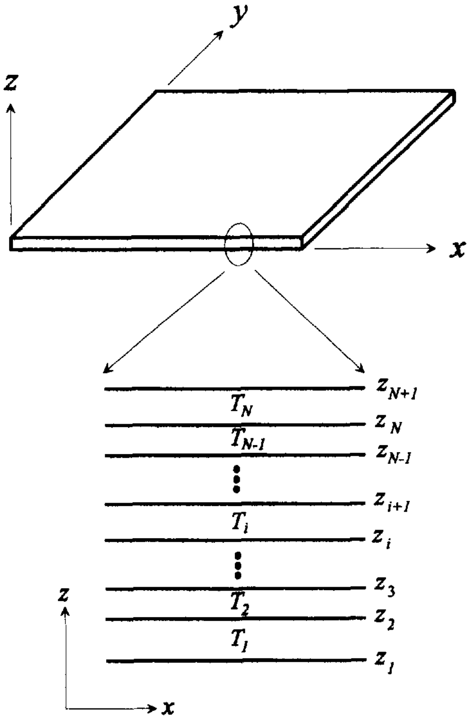

Fig. 1. An N-layer laminated composite with coordinate system and dimensions shown.

lamina is assumed to be homogeneous, isotropic, and thermoelastic. (An extension of our analysis to include the case where each lamina is orthotropic with one material coordinate coinciding with the z-coordinate is fairly straightforward; this is deferred to the Appendix.) Furthermore, the stress field may be quite general so long as the resulting heat conduction occurs only in the direction orthogonal to the layering, i.e., along the z-direction.

The inhomogeneous term in (3) is cast as a heat generation term for the j-th layer,

$$g_j(z,t) = -\alpha_j T_o \frac{\partial \sigma_{kk}}{\partial t}. \tag{22}$$

For each of the laminae, (3) may be written in the form

$$\frac{k_j}{C_j}\frac{\partial^2 T_j}{\partial z^2} + \frac{1}{C_j} g_j(z,t) = \frac{\partial T_j}{\partial t} \quad t > 0\,;\, z_j < z < z_{j+1}\,;\, j = 1, 2, 3, \ldots N. \tag{23}$$

The initial condition is $T_j(z,t) = T_o$ at $t = 0$ in $z_1 < z < z_{N+1}$. The outer boundaries are taken to be adiabatic. Then, the boundary and interface conditions are

$$\frac{\partial T_1}{\partial z} = 0 \quad \text{at } z = z_1 \tag{24}$$

$$-k_j \frac{\partial T_j}{\partial z} = f_{j+1}(T_j - T_{j+1}) \quad \text{at } z = z_{j+1}\,;\, j = 1, 2, \ldots, N-1 \tag{25}$$

$$-k_j \frac{\partial T_j}{\partial z} = -k_{j+1}\frac{\partial T_{j+1}}{\partial z} \quad \text{at } z = z_{j+1}\,;\, j = 1, 2, \ldots, N-1 \tag{26}$$

$$\frac{\partial T_N}{\partial z} = 0 \quad \text{at } z = z_{N+1} \tag{27}$$

where f_j is the film coefficient at the interfaces $z = z_j$. The finite value of the film coefficient in the interface condition, (25), represents a discontinuity of temperature at the interfaces. When $f_j \rightarrow \infty$, (25) reduces to (the familiar) $T_j = T_{j+1}$ at $z = z_{j+1}$, which represents the continuity of temperature or a perfect thermal contact.

As an aside we note that the concept of a film coefficient may be used not only to model a distinct imperfect interface but also an interphase layer which is sufficiently thin so that a linear temperature distribution across the interphase may be assumed. Then f may be thought of as the quotient of two parameters: (1) an effective interphase thickness, δ, and (2) an effective interphase thermal conductivity κ such that $f = \kappa/\delta$. Combining the interface conditions, (25) and (26), gives

$$-k_j \frac{\partial T_j}{\partial z} = -\kappa_{j+1}\frac{T_{j+1} - T_j}{\delta_{j+1}} = -k_{j+1}\frac{\partial T_{j+1}}{\partial z} \quad \text{at } z = z_{j+1}. \tag{28}$$

From this equation it can be clearly seen that the continuity of heat is maintained across an interphase layer whereas the temperature is discontinuous due to the finite distance and finite thermal conductivity of the interphase layer.

Following the standard integral transform techniques presented by Ozisik (1980, Ch. 14), the solution of the boundary value problem, (23) through (27), in terms of the fluctuating temperature field, was found to be

$$V_j(z) = -T_o \sum_{n=0}^{\infty} \frac{i\omega}{\beta_n^2 + i\omega} \frac{L_n}{N_n} \phi_{jn}(z) \tag{29}$$

where

$$L_n = \sum_{j=1}^{N} \alpha_j \int_{z_j}^{z_{j+1}} \sigma_{kk}(x, y, u)\phi_{jn}(u)\,\mathrm{d}u \tag{30}$$

and the norm N_n is given by

$$N_n = \sum_{j=1}^{N} C_j \int_{x_j}^{x_{j+1}} \phi_{jn}^2(u)\,\mathrm{d}u \tag{31}$$

where $\phi_{jn}(z)$ and β_n are, respectively, the eigenfunctions and eigenvalues of the following eigenvalue problem

$$\frac{\mathrm{d}^2\phi_{jn}}{\mathrm{d}z^2} + \frac{\beta_n^2}{(k/C)_j}\phi_{jn} = 0 \quad z_j < z < z_{j+1}\,; j = 1, 2, \ldots, N. \tag{32}$$

The subscript n implies that there is a countably infinite number of discrete eigenvalues and the corresponding eigenfunctions. The boundary conditions for (32) are the same as (24) through (27) but with T_j replaced by ϕ_{jn}. The eigenfunctions satisfy the following orthogonality relation

$$\sum_{j=1}^{N} C_j \int_{z_j}^{z_{j+1}} \phi_{jn}(u)\phi_{jm}(u)\,\mathrm{d}u = \begin{cases} 0 & n \neq m \\ N_n & n = m \end{cases} \tag{33}$$

where the norm N_n is defined in (31), and ϕ_{jn}, ϕ_{jm} represent two distinct eigenfunctions. A procedure for obtaining the eigenfunctions and eigenvalues is given by Ozisik (1980, Ch. 8) and, therefore, will not be described herein.

3.2. *Entropy produced*

An expression is now derived for the entropy produced per unit volume per cycle in the j-th layer, $\Delta s_j(z)$. Substituting the expression for V_j from (29) into (10), gives

$$\Delta s_j(z) = \pi k_j \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\omega(\omega^2 + \beta_n^2\beta_m^2)}{(\omega^2 + \beta_n^4)(\omega^2 + \beta_m^4)} \frac{L_n L_m}{N_n N_m} \frac{\mathrm{d}\phi_{jn}}{\mathrm{d}z} \frac{\mathrm{d}\phi_{jm}}{\mathrm{d}z}. \tag{34}$$

If the composite has any thermally *imperfect* interfaces, Bishop and Kinra (1996) have shown that an additional amount of entropy is produced *per unit area* per cycle, $\Delta s^{interface}$. For the special case of one-dimensional heat conduction under consideration, this entropy production per unit area for the j-th interface is given by

$$\Delta s_j^{interface} = -\frac{\pi}{\omega T_o^2}\left[k_j \,\mathrm{Re}\left(\frac{\mathrm{d}V_j}{\mathrm{d}z}\bar{V}_j\right)_{z=z_{j+1}^-} - k_{j+1}\,\mathrm{Re}\left(\frac{\mathrm{d}V_{j+1}}{\mathrm{d}z}\bar{V}_{j+1}\right)_{z=z_{j+1}^+}\right]. \tag{35}$$

The total entropy produced per cycle at all the interfaces, $\Delta s^{interface}$, is obtained by integrating over the area A of the interfaces

$$\Delta s^{interface} = \sum_{j=1}^{N-1} \int_A \Delta s_j^{interface} \, \mathrm{d}A. \tag{36}$$

Finally, the *total* entropy produced per cycle, ΔS, over the entire laminate is obtained by substituting (34) and (36) into (11).

3.3. *Work lost*

Substituting the expression for V_j from (29) into (17), we obtain the following expression for the work lost per unit volume per cycle in the j-th layer, $\Delta w_j(z)$.

$$\Delta w_j(z) = \pi T_o \alpha_j \sigma_{kk}(z) \sum_{n=0}^{\infty} \frac{\omega \beta_n^2}{\beta_n^4 + \omega^2} \frac{L_n}{N_n} \phi_{jn}(z). \tag{37}$$

3.4. *Elastothermodynamic damping*

An expression is now derived for the specific damping capacity, Ψ, of the entire laminated composite. In view of (11), (18), and (19), either (34) and (36) or (37) may be used to calculate the total work lost per cycle, ΔW. It turns out that the use of (37) results in a significant reduction in the algebraic effort. Accordingly, substituting (37) into (18) and using (20) and (21), we obtain

$$\Psi = \pi T_o \frac{\displaystyle\sum_{n=0}^{\infty} \frac{\omega \beta_n^2}{\beta_n^4 + \omega^2} \frac{\displaystyle\int_A L_n^2 \, \mathrm{d}A}{N_n}}{\displaystyle\int_R \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}R}. \tag{38}$$

We now have general expressions for the fluctuating temperature, V_j, entropy produced per unit volume per cycle Δs_j, work lost per unit volume per cycle Δw_j, and the specific damping capacity Ψ. This concludes the elastothermodynamic analysis of an N-layer laminated composite with thermally perfect or imperfect interfaces.

4. AN ILLUSTRATIVE EXAMPLE: A SYMMETRIC THREE-LAYER PLATE

As an example of the general procedure in Section 3, consider a symmetric three-layer plate with thermally *perfect* interfaces as shown in Figure 2. The externally applied time-harmonic moments are taken to be in phase with each other, and the factor $e^{i\omega t}$ is implied, as is customary. We assume that the deformations in the plate are adequately described by the classical (thin) laminate theory based on the well-known Kirchoff hypothesis (CLT). Since the plate is symmetric in both the material properties and the geometry, there is no coupling between bending and extension. Herein, we consider bending only. It is assumed that given the bending/twisting moments $M_x(x,y)$, $M_y(x,y)$, and $M_{xy}(x,y)$, the curvatures $\kappa_x(x,y)$, $\kappa_y(x,y)$, and $\kappa_{xy}(x,y)$, have been calculated using the CLT (see Jones, 1975, for example). Our sign convention is that positive moments produce positive curvatures (see Fig. 2); it is different from that of Jones (1975). With $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$, the nonzero stresses are given by

$$\sigma_{xx} = -\frac{E}{1-\nu^2}(\kappa_x + \nu\kappa_y)z, \quad \sigma_{yy} = -\frac{E}{1-\nu^2}(\kappa_y + \nu\kappa_x)z, \quad \sigma_{xy} = -\frac{E}{1+\nu}\kappa_{xy}z \tag{39}$$

so that

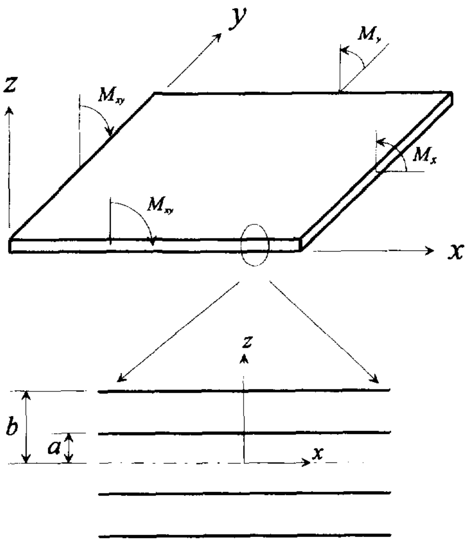

Fig. 2. A symmetric three-layer plate. Only the positive x and y faces are used to show the sign convention for the bending moments; the negative x and y faces are used to show the sign convention for the twisting moment.

$$\sigma_{kk} = -\frac{E}{1-\nu}(\kappa_x+\kappa_y)z. \tag{40}$$

Note that the hydrostatic stress field, σ_{kk}, is independent of the twist curvature, κ_{xy}, and, therefore, of the twisting moment, M_{xy}. In keeping with the assumptions in the CLT, the gradients of temperature in the x and y directions are very small compared to the gradients in the z-direction. From (8), the production of entropy depends upon the *square* of these temperature gradients. Therefore, for the purpose of calculating the total damping of the laminate, it appears reasonable to assume that heat conduction in the x and y directions may be neglected.

We now determine the fluctuating temperature field. Note that σ_{kk} in (40) is an odd function of z. Since the operator in (23) is symmetric in z, it follows that V is an odd function of z, and $V = 0$ at $z = 0$. Thus, it is sufficient to consider only the upper half of the plate, $z > 0$. In the following the subscript $()_1$ will refer to the inner layer of thickness $2a$, and the subscript $()_2$ will refer to the two outer layers each of thickness $(b-a)$; the total thickness of the plate is $2b$ (see Fig. 2). The volume fraction of the outer layers is $V_f \equiv 1-a/b$. Using the method of Ozisik (1980, Ch. 8), the eigenfunction for each of the laminae is given by

$$\phi_{1n}(z) = \sin(\gamma_n z/a) \tag{41}$$

$$\phi_{2n}(z) = A_n\cos(\eta_n z/b)+B_n\sin(\eta_n z/b) \tag{42}$$

where A_n and B_n are

$$A_n = -\frac{K\cos\gamma_n\cos\eta_n}{\sin\left(\eta_n\left(\frac{a}{b}-1\right)\right)}, \quad B_n = -\frac{K\cos\gamma_n\sin\eta_n}{\sin\left(\eta_n\left(\frac{a}{b}-1\right)\right)} \tag{43}$$

and

$$\gamma_n = \beta_n\sqrt{\tau_1},\quad \eta_n = \beta_n\frac{b}{b-a}\sqrt{\tau_2},\quad \tau_1 = \frac{C_1}{k_1}a^2,\quad \tau_2 = \frac{C_2}{k_2}(b-a)^2,\quad K = \sqrt{k_1C_1/k_2C_2}.$$

The constant τ is a characteristic time of heat conduction. We introduce a normalized frequency $\Omega_1 = \omega\tau_1$. The transcendental equation for the eigenvalues β_n is

$$\begin{vmatrix} -\sin\gamma_n & \cos(\eta_n a/b) & \sin(\eta_n a/b) \\ K\cos\gamma_n & \sin(\eta_n a/b) & -\cos(\eta_n a/b) \\ 0 & -\sin\eta_n & \cos\eta_n \end{vmatrix} = 0. \tag{44}$$

The expression for V_j given by eqn (29) may now be cast in the following form

$$V_j(z) = -T_o b(\kappa_x+\kappa_y)\left(\frac{E}{1-\nu}\frac{\alpha}{C}\right)_1 \sum_{n=0}^{\infty}\left(\frac{\Omega_1^2+i\Omega_1\gamma_n^2}{\Omega_1^2+\gamma_n^4}\right)R_n\phi_{jn}(z)\quad j=1,2 \tag{45}$$

where

$$R_n = \frac{(a/b)^2 I_1 + \dfrac{E_2}{E_1}\dfrac{1-\nu_1}{1-\nu_2}\dfrac{\alpha_2}{\alpha_1}I_2}{(a/b)I_3+(C_2/C_1)I_4} \tag{46}$$

$$\begin{aligned} I_1 &= \frac{1}{\gamma_n^2}\sin(\gamma_n) - \frac{1}{\gamma_n}\cos(\gamma_n) \\ I_2 &= \frac{1}{\eta_n}\left\{A_n\left[\frac{1}{\eta_n}(\cos(\eta_n)-\cos(\eta_n a/b)) + \sin(\eta_n) - \frac{a}{b}\sin(\eta_n a/b)\right]\right. \\ &\quad \left. + B_n\left[\frac{1}{\eta_n}(\sin(\eta_n)-\sin(\eta_n a/b)) - \cos(\eta_n) + \frac{a}{b}\cos(\eta_n a/b)\right]\right\} \\ I_3 &= \frac{1}{2} - \frac{1}{4\gamma_n}\sin(2\gamma_n) \\ I_4 &= \frac{1}{2}\left(1-\frac{a}{b}\right)(A_n^2+B_n^2) + \frac{1}{4\eta_n}(A_n^2-B_n^2)[\sin(2\eta_n)-\sin(2\eta_n a/b)] \\ &\quad - \frac{1}{2\eta_n}A_nB_n[\cos(2\eta_n)-\cos(2\eta_n a/b)]. \end{aligned} \tag{47}$$

Intuitively, as $\Omega_1 \to 0$ the time period of vibration goes to infinity, during each cycle heat has plenty of time to diffuse between the "cold" tensile side and the "hot" compressional side, the isothermal conditions are obtained; therefore, $V(z) \to 0$, $-b \leqslant z \leqslant b$. In the other extreme, as $\Omega_1 \to \infty$, the time period of vibration goes to zero, and heat has very little time to diffuse before the stress (and therefore the direction of heat conduction) is reversed. Therefore, with the exception of the immediate vicinity of the interface where there will always be some heat conduction no matter how high the frequency, essentially adiabatic conditions are obtained. Under strictly adiabatic conditions (i.e., when we set the thermal conductivity $k = 0$), the temperature in each layer is given merely by the thermoelastic effect, (1). Using (40),

$$V_j(z) = T_o(\kappa_x+\kappa_y)\left(\frac{E}{1-\nu}\frac{\alpha}{C}\right)_j z,\quad j=1,2. \tag{48}$$

As a check on our algebra, we took care to verify that (45) satisfies these two conditions.

We now introduce a normalized fluctuating temperature as

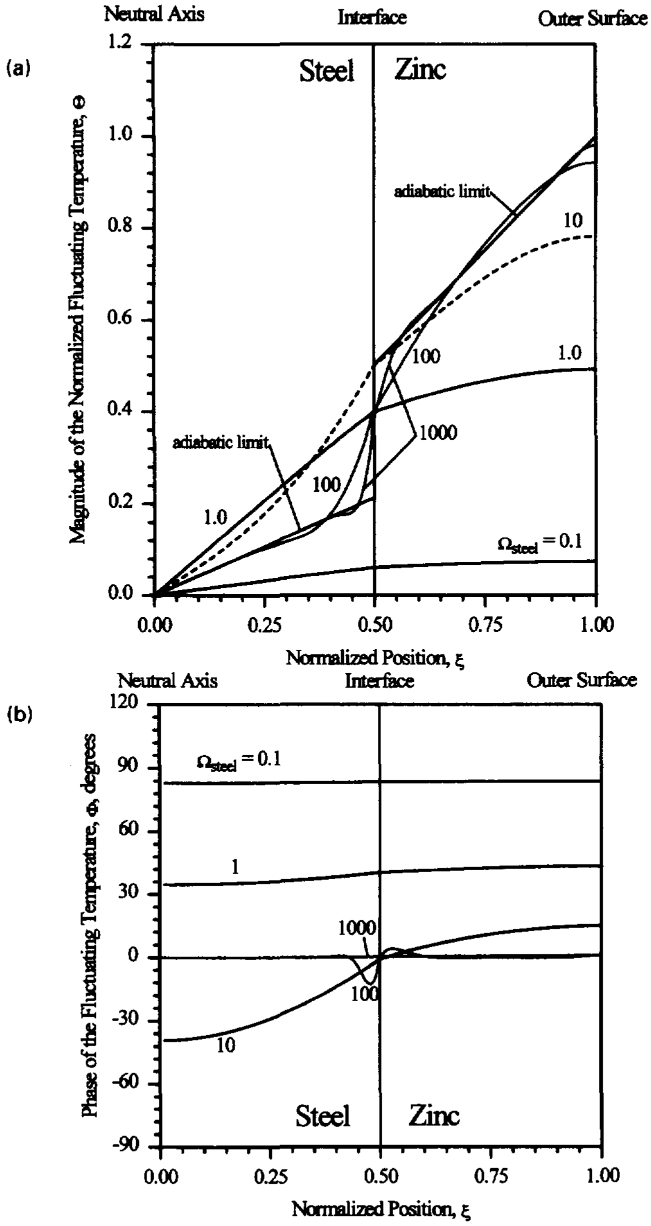

Fig. 3. (a) Magnitude and (b) phase of the normalized fluctuating temperature vs the normalized position for the symmetric three-layer, zinc-steel-zinc plate with thermally *perfect* interfaces, for $\Omega_{steel} = 0.1$, 1, 10, 100, and 1000, and for a zinc volume fraction $V_f = 0.5$.

$$\Theta = \frac{V}{T_o b(\kappa_x + \kappa_y)\left(\dfrac{E}{1-\nu}\dfrac{\alpha}{C}\right)_2}. \tag{49}$$

A physical feel for the denominator may be developed as follows. If a *homogeneous* Kirchoff plate of thickness $2b$ is subjected, under adiabatic conditions, to static moments (M_x, M_y) with resulting adiabatic curvatures (κ_x, κ_y), then the increase in temperature of the upper surface will be $T_o b(\kappa_x + \kappa_y)(E\alpha/(1-\nu)C)$. In polar form $\Theta = \|\Theta\| \exp(i\Theta)$, where $\|\Theta\|$ is the magnitude and Φ is the phase of Θ. In Figs 3(a) and (b) we have plotted $\|\Theta\|$ and Φ, respectively, vs the normalized position, $\xi = z/b$, for a zinc-steel-zinc plate with $\Omega_1 = \Omega_{steel}$ as a parameter, and a zinc volume fraction $V_f = 0.5$. (One of the potential engineering applications of high-damping zinc-coated steel sheets is in the design of automobile bodies for a quiet ride.) Recall, at $z = 0$, $V \equiv 0$ at all frequencies. Note that for $\Omega_{steel} = 0.1$

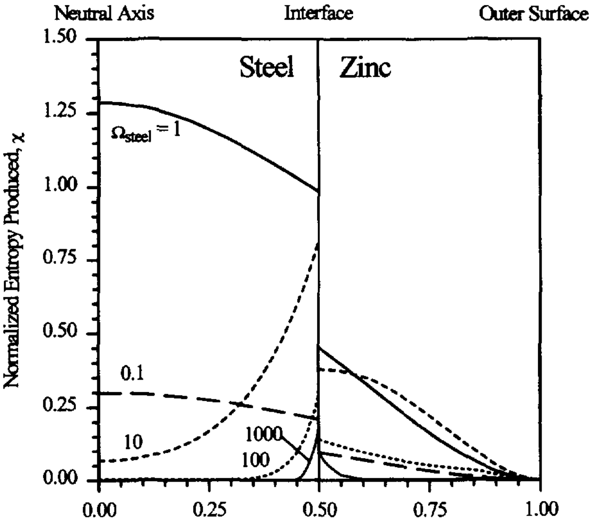

Fig. 4. Normalized entropy vs the normalized position for the symmetric three-layer, zinc-steel-zinc plate with thermally *perfect* interfaces, for $\Omega_{steel} = 0.1$, 1, 10, 100, and 1000, and for a zinc volume fraction $V_f = 0.5$.

the conditions are essentially isothermal, $\|\Theta\|$ is approaching zero, and the fluctuating temperature is out of phase with the applied stress by approximately $\pi/2$. As Ω_{steel} increases, temperature gradients develop throughout the thickness of the plate, and spatial variations in Φ become increasingly localized near the interface. For reference, the temperature given by (48) is plotted as the "adiabatic limit" (straight lines) in Fig. 3(a). At very high frequencies, $\Omega_{steel} = 1000$, $\|\Theta\|$ approaches the adiabatic limit, and $\phi \to 0$, with the obvious exception of the immediate vicinity of the interface as discussed previously.

Using (34) the entropy produced per cycle per unit volume in each layer, Δs_j, is given by

$$\Delta s_j(z) = \pi k_j \tau_1 \left(\frac{E}{1-\nu}\frac{\alpha}{C}\right)_1^2 b^2(\kappa_x+\kappa_y)^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Omega_1(\Omega_1^2+\gamma_n^2\gamma_m^2)}{(\Omega_1^2+\gamma_n^4)(\Omega_1^2+\gamma_m^4)} R_n R_m \frac{\mathrm{d}\phi_{jn}}{\mathrm{d}z}\frac{\mathrm{d}\phi_{jm}}{\mathrm{d}z}, \quad j = 1, 2. \tag{50}$$

We introduce a normalized measure of entropy produced,

$$\chi = \frac{\Delta s}{(E_2\alpha_2 b(\kappa_x+\kappa_y))^2/(C_2(1-\nu_2)^2)}. \tag{51}$$

A physical feel for the denominator may be developed as follows. If a homogeneous Kirchoff plate of thickness $2b$ at an equilibrium temperature T_o is subjected, under adiabatic conditions, to static moments (M_x, M_y) with resulting adiabatic curvatures (κ_x, κ_y), then $V = T_o(\kappa_x+\kappa_y)(E\alpha/(1-\nu)C)z$. While holding the moments fixed, the plate is allowed to return to its equilibrium temperature T_o by way of *internal* heat conduction; there is no heat exchange with the surroundings. Next, the plate is unloaded adiabatically; then $V = -T_o(\kappa_x+\kappa_y)(E\alpha/(1-\nu)C)z$. Once again, the plate is allowed to return to its equilibrium temperature T_o by way of internal heat conduction. The *total* entropy produced *per unit area* during this complete cycle is $((\kappa_x+\kappa_y)bE\alpha)^2 b/3(1-\nu)^2 C$. For use in the next paragraph, the *total* external work lost during this cycle per unit *area* is $((\kappa_x+\kappa_y)bE\alpha)^2 bT_o/3(1-\nu)^2 C$.

The normalized entropy is plotted in Fig. 4 vs the normalized position ξ for the zinc-steel-zinc plate, with Ω_{steel} as a parameter, and a zinc volume fraction $V_f = 0.5$. As expected from the adiabatic boundary conditions, the entropy produced is zero at $\xi = 1$. At low

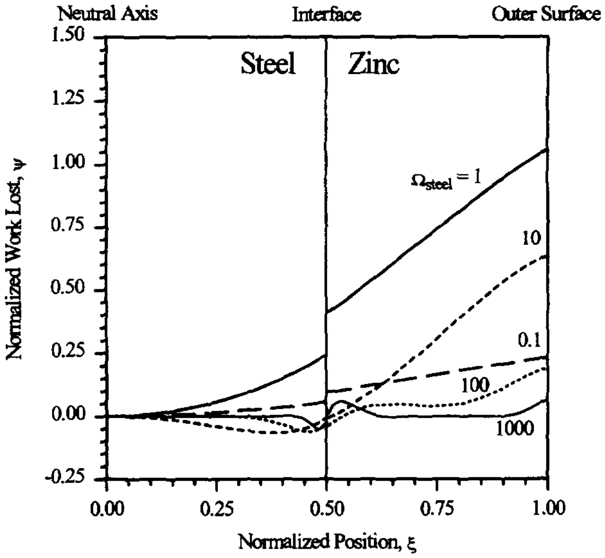

Fig. 5. Normalized work lost vs the normalized position for the symmetric three-layer, zinc-steel-zinc plate with thermally *perfect* interfaces, for $\Omega_{steel} = 0.1$, 1, 10, 100, and 1000, and for a zinc volume fraction $V_f = 0.5$.

frequencies (e.g., $\Omega_{steel} = 0.1$) the problem is essentially isothermal, and accordingly, the entropy produced is approaching zero throughout the plate. At high frequencies (e.g., $\Omega_{steel} = 1000$) the problem is essentially adiabatic and, with the exception of the immediate vicinity of the interface, the entropy produced is nearly zero.

Using (37) the work lost per unit volume per cycle in each layer, Δw_j, is given by

$$\Delta w_j(z) = \pi\left(\frac{\alpha E}{1-v}\right)_j T_o b^2(\kappa_x+\kappa_y)^2 \frac{z}{b}\left(\frac{E}{1-v}\frac{\alpha}{C}\right)_1 \sum_{n=0}^{\infty}\frac{\Omega_1\gamma_n^2}{\Omega_1^2+\gamma_n^4}R_n\phi_{jn}(z), \quad j=1,2. \tag{52}$$

We introduce a normalized measure of work lost,

$$\psi = \frac{\Delta w}{(E_2\alpha_2 b(\kappa_x+\kappa_y))^2 T_o/(C_2(1-v_2)^2)}. \tag{53}$$

A physical feel for the denominator was discussed in the context of (51). The normalized work lost is plotted in Fig. 5 for the zinc-steel-zinc plate vs the normalized position ξ with Ω_{steel} as a parameter, and a zinc volume fraction $V_f = 0.5$. Note that the normalization of Δw and Δs was done in such a manner that setting $\psi = \chi$ corresponds to $\Delta w = T_o\Delta s$, i.e., one unit of normalized entropy produced corresponds to one unit of normalized work lost. Accordingly, ψ (Fig. 5) and χ (Fig. 4) are plotted on an identical scale. In accordance with the results of Bishop and Kinra (1996), pointwise $\psi \neq \chi$ or $\Delta w \neq T_o\Delta s$. Comparing Figs 4 and 5 we see that ψ is maximum at the outer surface $\xi = 1$ whereas χ is identically zero. Conversely, at the neutral axis ψ is zero for all frequencies, whereas χ is not. There are regions where ψ becomes *negative*. At first sight this may appear disturbing for it implies that work is being produced rather than converted into heat. However, the only constraint imposed by the second law of thermodynamics is that the *total* entropy produced throughout the plate (a closed thermodynamic system which does not exchange heat with its surroundings) during one cycle of loading is either zero or positive, i.e., $\Delta S \geqslant 0$. It follows from (19) that $T_o\Delta S \equiv \Delta W \geqslant 0$, i.e., the *total* work lost throughout the plate during one

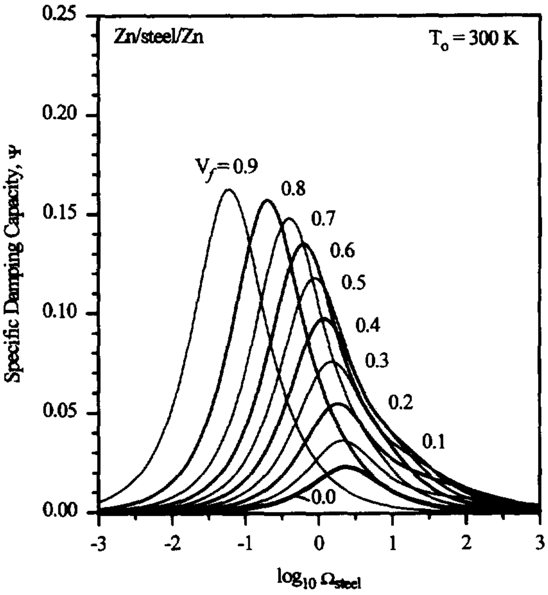

Fig. 6. Specific damping capacity vs the normalized frequency Ω_{steel} for the symmetric three-layer, zinc-steel-zinc plate with thermally *perfect* interfaces, with zinc volume fraction V_f as the parameter.

cycle should be zero or positive. It then follows that $\Psi \geqslant 0$. From Fig. 6, it is obvious that this second-law constraint is clearly satisfied. Moreover, negative values of ψ have in the past been observed in the case of a perfect interface between two semi-infinite rods of thermoelastically different materials (Kinra and Milligan, 1994) and a Griffith crack (Kinra and Bishop, 1995).

Using (38) the specific damping capacity for the three-layer composite plate is given by

$$\Psi = \Psi_{ol} \frac{\sum_{n=1}^{\infty} \frac{\Omega_1 \gamma_n^2}{\Omega_1^2 + \gamma_n^4} Q_n}{\frac{1}{3}\left[\frac{1-\nu_1}{1+\nu_1}(a/b)^3 \frac{f_1(\kappa)}{f_3(\kappa)} + \frac{E_2}{E_1}\frac{(1-\nu_1)^2}{1-\nu_2^2}(1-(a/b)^3)\frac{f_2(\kappa)}{f_3(\kappa)}\right]} \tag{54}$$

where Q_n is defined as

$$Q_n = \frac{\left[(a/b)^2 I_1 + \frac{E_2}{E_1}\frac{1-\nu_1}{1-\nu_2}\frac{\alpha_2}{\alpha_1} I_2\right]^2}{\frac{a}{b} I_3 + \frac{C_2}{C_1} I_4} \tag{55}$$

and

$$f_j(\kappa) = \int_{xy} [\kappa_x^2 + \kappa_y^2 + 2(1-\nu_j)\kappa_{xy}^2 + 2\nu_j \kappa_x \kappa_y]\,\mathrm{d}x\,\mathrm{d}y, \quad j = 1, 2$$

$$f_3(\kappa) = \int_{xy} (\kappa_x + \kappa_y)^2\,\mathrm{d}x\,\mathrm{d}y. \tag{56}$$

A modulus of elastothermodynamic damping has been defined as $\Psi_o = 2\pi E\alpha^2 T_o/C$. This parameter first appeared in Zener's work (1937) when he analyzed the thermoelastic damping of a homogeneous beam undergoing flexural vibrations. Zener called it the relaxation

Table 1. Thermoelastic constants (nominal values)

	E (GPa)	k ($\mathrm{Js^{-1}\,m^{-1}\,K^{-1}}$)	$\alpha\,(10^{-6})$ ($\mathrm{K^{-1}}$)	$C(10^6)$ ($\mathrm{J\,m^{-3}\,K^{-1}}$)	ν	Ψ_o
Al	70	220	24	2.4	0.33	0.030
Steel	200	52	12	3.8	0.33	0.015
Ti	120	39	8.4	2.3	0.33	0.0066
Zn	100	110	40	2.7	0.33	0.11
Ni	210	92	13	3.9	0.31	0.018

strength. In honor of Zener, we would like to rename it as the Zener's modulus of elastothermodynamic damping (cf. the Young's modulus of elasticity). This parameter is listed in Table 1.

The damping of the zinc-steel-zinc composite plate is plotted in Fig. 6 as a function of Ω_{steel} for various values of V_f (the volume fraction of zinc), with the curvatures $\kappa_{xy} = 0$, $\kappa_x = \kappa_y =$ constant, and $T_o = 300$ K. For a given value of V_f, as $\Omega_{steel} \to 0$ essentially isothermal conditions are obtained, and $\Psi \to 0$. At the other extreme, as $\Omega_{steel} \to \infty$, away from the interfaces, essentially adiabatic conditions are obtained, and, once again $\Psi \to 0$. For comparison, Ψ for an uncoated steel plate ($V_f = 0.0$) is also shown; the maximum damping is $\Psi_{max} \cong 0.025$. At a "very small cost" of adding 20% zinc (by volume), Ψ_{max} increases to 0.055, i.e., by about 100%. When $V_f = 0.4$, $\Psi_{max} \cong 0.1$, i.e., an increase of about 400%. (We note that for other material combinations, these damping values may be much smaller.)

The special case of a homogeneous plate may be obtained from (54) by letting $V_f \to 0$. After some careful analysis, we obtained the following result

$$\Psi = \frac{1+\nu f_3(\kappa)}{1-\nu f_1(\kappa)}\left[\Psi_o \frac{96}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{\Omega}{\Omega^2+(2n+1)^4}\right] \tag{57}$$

where we have defined an *unsubscripted* normalized frequency, $\Omega = \omega C(2a)^2/\pi^2 k$, so that our result will be directly comparable to a classic result by Zener (1937). Note that $2a$ is the thickness of the homogeneous plate. The expression in the square brackets in (57) is precisely the damping of a homogeneous Euler-Bernoulli beam subjected to in-plane bending (Zener, 1937). Thus, the following elegantly simple result follows immediately: for a *homogeneous plate*.

$$\Psi_{plate} = \lambda \Psi_{beam} \tag{58}$$

where $\lambda = (1+\nu) f_3(\kappa)/(1-\nu) f_1(\kappa)$. Consider the special case, $\kappa_{xy} = 0$, $\kappa_x = \kappa_y =$ constant, and $\nu = 1/3$; then $\lambda = 3$. Therefore, $\Psi_{plate} = 3\Psi_{beam}$, i.e., the damping in a plate subjected to equal biaxial curvatures is three times the damping in a beam subjected to the same curvature in the plane of the beam. Furthermore, if $\kappa_{xy} = \kappa_y = 0$ then $\lambda = 2$, i.e., when subjected to identical curvatures, the plate damping is twice the beam damping. Zener (1937) made a clever observation that the series in (57) is accurately represented by merely its first term ($n = 0$), and that $96/\pi^4 = 0.9855 \cong 1$. Then, (57) reduces to the well-known form for a standard anelastic solid

$$\Psi = \lambda\left[\Psi_o \frac{\Omega}{1+\Omega^2}\right] \quad \text{(homogeneous plate).} \tag{59}$$

Equation (59) also affords us a simple setting in which we can build our intuition concerning Ω. To this end we first introduce the notion of a diffusion length $l = (2k/\omega C)^{1/2}$ so that $\Omega = (8/\pi^2)(a/l)^2$. A physical feel for the diffusion length may be obtained as follows. Consider a semi-infinite rod occupying $0 \leqslant x < \infty$ at an equilibrium temperature T_o, subjected to a time-harmonic temperature $V = V_o e^{i\omega t}$ at the end $x = 0$. In the interior the temperature variation is given by $V(x) = V_o \exp(-(1+i)x/l)$ i.e., the diffusion length controls the rate of spatial decay. Thus, at $x = l$, $|V/V_o| = 1/e$, i.e., at a depth of one diffusion length the temperature drops by $1/e$. Returning to eqn (59) we observe that the maximum damping is $\Psi_{max} = \lambda\Psi_o/2$ and occurs at $\Omega = 1$ which translates to $(a/l) \cong 1$. We thus gain an important insight into the physics of the problem under consideration: for a homogeneous plate the maximum damping occurs when the diffusion length is comparable to one-half of the plate thickness. A parallel insight can be gained by thinking in terms of the characteristic time $\tau = a^2C/k$. Then $\Omega = 1 \Leftrightarrow f\tau = \pi/8$, where $f = \omega/2\pi$ is the frequency in Hertz. Taking $8/\pi \cong 2, f = 1/2\tau$. Viewing τ a "the time taken by heat to travel (or diffuse across) a distance 'a'," 2τ becomes the round-trip travel time. In mechanical vibrations, for example, the longitudinal vibrations of a rod of length a, the fundamental resonance occurs when the frequency $f = 1/2\tau$, where 2τ is the round-trip travel time of the wave, $\tau = a/c$, and c is the longitudinal wave velocity. In view of this strong analogy, it is tempting to view $\Omega = 1$ as the excitation of a thermal resonance which results in maximum damping.

Finally, the Office of Naval Research recently concluded an Accelerated Research Initiative (ARI) in which the objective was to design materials with $\Psi \geqslant 0.06$. Measurement of damping in continuous *fiber-reinforced* and *particle-reinforced* metal-matrix composites has been reported by several researchers (Timmerman, 1982; Timmerman and Doherty, 1984; Steckel, 1985; Steckel and Nelson, 1985; Bhagat *et al.*, 1989; Kinra *et al.*, 1991; Kinra and Wren, 1992; Wren and Kinra, 1992; Zhang *et al.*, 1992; Perez *et al.*, 1993; Riviere and Woirgard, 1993). Typically, Ψ was found to be of the order of a few percent. (As far as we know, similar data concerning the damping of *laminated* MMC's has not been reported in the open literature.) The measured damping is due to *all* relaxation mechanisms including elastothermodynamic relaxation. We have shown that through a proper choice of constituent materials, laminae size, and frequency, our analysis can be used to *design* high-damping laminated composites in which elastothermodynamic damping is the dominant component of the total damping. Consider, for example, a one mm thick steel plate with a zinc volume fraction, $V_f = 0.5$. The highest damping is $\Psi_{max} = 0.12$, which is quite high. Moreover, a significant damping (say, $\Psi > 0.01$) is obtained over a broad range (more than three decades) of frequencies, $0.1 < f < 200$ Hz, which has applications in absorbing low-frequency noise. For comparison, it is well known that the combined *intrinsic* material damping due to all other relaxation mechanisms in most metals is generally very small, $\Psi < 0.01$. Moreover, the elastothermodynamic damping is in addition to the damping due to all other mechanisms.

5. CONCLUSIONS

Analytical expressions for the temperature field, entropy produced, work lost, and the elastothermodynamic damping for an N-layer laminated composite with perfect or imperfect thermal interfaces have been obtained. The composite plate may be subjected to any stress field so long as the resulting heat conduction may be described by a single spatial coordinate orthogonal to the layering. As an illustrative example, results were presented for a symmetric three-layer plate with *perfect* thermal interfaces subjected to biaxial bending. Numerical results were presented for a zinc-steel-zinc composite plate. Through a proper choice of constituent materials, laminae size, and frequency, our analysis can be used to design laminated composites in which the elastothermodynamic damping is a significant fraction of the total damping.

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REFERENCES

Alblas, J. (1961) On the general theory of thermo-elastic friction. *Applied Scientific Research* **10,** 349–362.
Alblas, J. (1981) A note on the theory of thermoelastic damping. *Journal of Thermal Stresses* **4,** 333–355.
Bejan, A. (1996) *Entropy Generation Minimization*. CRC Press, New York.

Bhagat, R., Amateau, M., Conway, J., Paulick, J., Chisholm, J., Parnell, J. and Seidensticker, D. (1989) Squeeze cast metal matrix composites: evaluation of their strength, damping capacity and corrosion resistance. *Journal of Composite Materials* **23,** 961–975.

Bhagat, R., Amateau, M. and Smith, E. (1989) Damping behavior of planar random carbon fiber reinforced 6061 Al matrix composites fabricated by high-pressure infiltration casting. *Journal of Composites Technology and Research* **11,** 113–116.

Biot, M. (1956) Thermoelasticity and irreversible thermodynamics. *Journal of Applied Physics* **27,** 240–253.

Bishop, J. and Kinra, V. (1993) Thermoelastic damping of a laminated beam in flexure and extension. *Journal of Reinforced Plastics and Composites* **12,** 210–226.

Bishop, J. and Kinra, V. (1994) Elastothermodynamic damping in composite materials. *Mechanics of Composite Materials and Structures* **1,** 75–93.

Bishop, J. and Kinra, V. (1995) Elastothermodynamic damping in particle-reinforced matal-matrix composites. *Metallurgical and Materials Transactions* **26A,** 2773–2783.

Bishop, J. and Kinra, V. (1996) Equivalence of the mechanical and entropic descriptions of elastothermodynamic damping in composite materials. *Mechanics of Composite Materials and Structures* **3,** 83–95.

Boley, B. and Weiner, J. (1960) *Theory of Thermal Stresses.* John Wiley & Sons, New York.

Brillouin, L. (1953) *Wave Propagation in Periodic Structures.* Dover, New York.

Chadwick, P. (1962) Thermal damping of a vibrating elastic body. *Mathematika* **9,** 38–48.

De Batist, R. (1972) *Internal Friction of Structural Defects in Crystalline Solids.* American Elsevier, New York.

de Groot, S. and Mazur, P. (1984) *Non-Equilibrium Thermodynamics.* Dover, New York.

Deresiewicz, H. (1957) Plane waves in a thermoelastic solid. *Journal of the Acoustical Society of America* **29,** 204–209.

Eshelby, J. (1949) Dislocations as a cause of mechanical damping in metals. *Proceedings of the Royal Society London* **A197,** 396–416.

Gillis, W. (1968) Damping of thermoelastic structures. *Technical Memorandum X*-53722, George C. Marshall Space Flight Center, Hunstville, Alabama.

Goodman, L., Chang, C. and Robinson, A. (1962) Thermoelastic damping. *Technical Documentary Report No. ASD-TDR*-62-1031, Wright-Patterson Air Force Base, Ohio.

Jones, R. M. (1975) *Mechanics of Composite Materials.* Hemisphere, New York.

Kinra, V. and Bishop, J. (1996) Elastothermodynamic analysis of a Griffith crack. *Journal of the Mechanics and Physics of Solids* **44,** 1305–1336.

Kinra, V. and Milligan, K. (1994) A second law analysis of thermoelastic damping. *ASME Journal of Applied Mechanics* **61,** 71–76.

Kinra, V. and Wolfenden, A. (eds.) (1992) *M^3D: Mechanics and Mechanisms of Material Damping, ASTM STP* 1169, American Society for Testing and Materials, Philadelphia, Pennsylvania.

Kinra, V., Wren, G., Rawal, S. and Misra, M. (1991) On the influence of ply-angle on damping and modulus of elasticity of a metal-matrix composite. *Metallurgical Transactions* **22A,** 641–651.

Kinra, V. and Wren, G. (1992) Axial damping in metal-matrix composites. I: a new technique for measuring phase difference to 10^{-4} radians. *Experimental Mechanics* **32,** 163–171.

Landau, L. and Lifshitz, E. (1986) *Theory of Elasticity.* Pergamon Press, NY.

Lee, U. (1985) Thermoelastic and electromagnetic damping analysis. *AIAA Journal* **23,** 1783–1790.

Lücke, K. (1956) Ultrasonic attenuation caused by thermoelastic heat flow. *Journal of Applied Physics* **27,** 1433–1438.

Milligan, K. and Kinra, V. (1993) On the damping of a one-dimensional inclusion in a uniaxial bar. *Mechanics Research Communications* **20,** 137–142.

Milligan, K. and Kinra, V. (1995a) Elastothermodynamic damping of fiber-reinforced metal-matrix composites. *ASME Journal of Applied Mechanics* **62,** 441–449.

Milligan, K. and Kinra, V. (1995b) A numerical investigation of the elastothermodynamic damping of fiber-reinforced metal-matrix composites. *ASME Journal of Applied Mechanics* **62,** 816–817.

Nowick, A. and Berry, B. (1972) *Anelastic Relaxation in Crystalline Solids.* Academic Press, New York.

Nowinski, J. (1978) *Theory of Thermoelasticity with Applications.* Sijthoff and Noordhoff International Publ. B.V., Alphen Aan Den Rijn, The Netherlands.

Ozisik, M. (1980) *Heat Conduction.* John Wiley & Sons, New York.

Perez, R., Zhang, J., Gungor, M. and Lavernia, E. (1993) Damping behavior of 6061 Al/Gr metal matrix composites. *Metallurgical Transactions A* **24A,** 701–712.

Riviere, A. and Woirgard, J. (1993) Internal friction in metal matrix composites. In *Damping of Multiphase Inorganic Materials* (ed. R. Bhagat). ASM International, Materials Park, OH, 1993, pp. 25–28.

Riviere, A., Gadaud, P. and Woirgard, J. (1993) Damping in glass ceramic matrix composites. In *Damping of Multiphase Inorganic Materials* (ed. R. Bhagat). ASM International, Materials Park, OH, 1993, pp. 137–140.

Shieh, R. (1971) Thermoelastic damping and its effect on flutter of stressed panels situated in a supersonic airflow. *NASA TND*-6448, NASA Langley Research Center, Hampton, Virginia, 1971.

Shieh, R. (1975) Thermoelastic vibration and damping for circular timoshenko beams. *ASME Journal of Applied Mechanics* **42,** 405–410.

Shieh, R. (1979) Eigensolutions for coupled thermoelastic vibrations of timoshenko beams. *ASME Journal of Applied Mechanics* **46,** 169–174.

Steckel, G. (1985) Material damping behavior of Gr/Mg composites. *Aerospace Corporation ONR Report Number TOR*-0084-*A*-5726-01(-1), The Aerospace Corporation, El Segundo, CA.

Steckel, G. and Nelson, B. (1985) Mechanical damping behavior of graphite-magnesium. *Aerospace Corporation ONR Report Number TOR*-0086(6726-01)-1, The Aerospace Corporation, El Segundo, CA.

Tasi, J. (1963) Thermoelastic dissipation in vibrating plates. *ASME Journal of Applied Mechanics* **30,** 562–567.

Tasi, J. and Herrmann, G. (1964) Thermoelastic dissipation in high-frequency vibrations of crystal plates. *Journal of the Acoustical Society of America* **36,** 100–110.

Timmerman, N. (1982) Damping characteristics of metal matrix composites. Army Materials and Mechanics Research Center Report, *AMMRC-TR*-82-19.

Timmerman, N. and Doherty, J. (1984) Loss factors in metal matrix composites. Army Materials and Mechanics Research Center Report, *AMMRC-TR*-84-22.

Wolfenden, A. and Wolla, J. (1996) Dynamic mechanical properties. In *Metal Matrix Composites*. Academic Press, New York, pp. 287–328.

Wren, G. and Kinra, V. (1992) Axial damping in metal-matrix composites. II: a theoretical model and its experimental verification. *Experimental Mechanics* **32,** 172–178.

Zemansky, M. and Dittman, R. (1981) *Heat and Thermodynamics*, 6th ed. McGraw-Hill, New York, New York, pp. 199–206.

Zener, C. (1937) Internal friction in solids I. Theory of internal friction in reeds. *Physical Review* **52,** 230–235.

Zener, C. (1938) Internal friction in solids II. General theory of thermoelastic internal friction. *Physical Review* **53,** 90–99.

Zhang, J., Perez, R. and Lavernia, E. (1992) Damping characterization of graphite particulate reinforced aluminum composites. In *Developments in Ceramic and Metal-Matrix Composites* (ed. K. Upadhya). TMS 1992, San Diego, CA.

APPENDIX: ORTHOTROPIC LAMINAE

The analysis presented in this paper is extended to include the case where each of the laminae is orthotropic with one material coordinate coinciding with the z-axis, e.g., continuous-fiber-reinforced composite laminates. We will assume that for a given loading, the CLT has been used to calculate the stress field, $\sigma_{lm}(x, y, z)$, throughout the laminate. The essential difference between an orthotropic and an isotropic lamina is that now the coefficient of thermal expansion (α_{lm}) and the thermal conductivity (k_{lm}) must be represented by tensors. In this Appendix, we present only those equations that change in the course of this generalization. For the convenience of the reader, the equations in the Appendix are numbered to correspond to their isotropic version in the body of the paper. Equations (1), (3), (8), (10), (16), (17) become, respectively,

$$\left(\frac{\partial T}{\partial \sigma_{lm}}\right)_S = -T\frac{\alpha_{lm}}{C} \tag{A1}$$

$$C\frac{\partial T}{\partial t} = (k_{lm}T_{,m})_{,l} - T_o\alpha_{lm}\frac{\partial \sigma_{lm}}{\partial t} \tag{A2}$$

$$\dot{s}_p = \frac{1}{T_o^2}k_{lm}T_{,l}T_{,m} \tag{A3}$$

$$\Delta s(\mathbf{x}, \omega) = \frac{\pi}{\omega T_o^2}k_{lm}\,\mathrm{Re}(V_{,l}\bar{V}_{,m}) \tag{A4}$$

$$\varepsilon_{lm}^{th}(\mathbf{x}, \omega) = \alpha_{lm}V(\mathbf{x}, \omega) \tag{A5}$$

$$\Delta w(\mathbf{x}, \omega) = -\pi\alpha_{lm}\sigma_{lm}(\mathbf{x})\,\mathrm{Im}\,V(\mathbf{x}, \omega). \tag{A6}$$

We now turn to Section 3. Recall that it is assumed that heat conduction occurs only in the z-direction. For brevity, let k_j be the thermal conductivity of the jth lamina in the z-direction, i.e., $k_j = k_{33}^j$. The analysis in Section 3 still remains valid provided we make the following three changes. Equation (22) becomes

$$g_j(z, t) = -T_o\alpha_{lm}^j\frac{\partial \sigma_{lm}}{\partial t}. \tag{A7}$$

(Note that only k, l, and m, are tensorial indices, but j is not.) The L_n in eqn (29) is now given by

$$L_n = \sum_{j=1}^{N}\int_{z_j}^{z_{j+1}} \alpha_{lm}^j\sigma_{lm}(x, y, u)\phi_{jn}(u)\,\mathrm{d}u \tag{A8}$$

where the j-superscript on α_{lm} denotes the j-th lamina as usual. Finally, eqn (37) becomes

$$\Delta w_j(z) = \pi T_o\alpha_{lm}^j\sigma_{lm}(z)\sum_{n=0}^{\infty}\frac{\omega\beta_n^2}{\beta_n^4+\omega^2}\frac{L_n}{N_n}\phi_{jn}(z) \tag{A9}$$

and Ψ is still given by eqn (38).